Philosophy 211 – Problems from class Nov 9th and 14th

On Nov 9th, we were discussing properties of relations such as symmetry – $\forall x \forall y(Rxy \rightarrow Ryx)$ – when I said "If a relation R is symmetric and if its R then its S, then S is also symmetric." Here is the problem I wrote up:

$$\forall x \forall y(Rxy \rightarrow Ryx),\ \forall x \forall y(Rxy \rightarrow Sxy) \vdash \forall x \forall y(Sxy \rightarrow Syx)$$

Soon after starting the problem I realized that it couldn't be done. This is because we need the second premise to be stronger. This is the correct problem:

$$\forall x \forall y(Rxy \rightarrow Ryx),\ \forall x \forall y(Rxy \leftrightarrow Sxy) \vdash \forall x \forall y(Sxy \rightarrow Syx)$$

Notice the difference in the second premise. It can't just be that IF a pair is R then its S, R and S have to be equivalent. I very quickly went through an outline of the proof, but here it is in full:

Step 1: To prove a universal claim, prove an arbitrary instance of it. Lets use 'a' to replace 'x' in that instance. This is also a universal claim so I will try to prove an arbitrary instance of it. Here, I cannot the 'y' by 'a' so I will choose a different name. Let's use 'b'. Then we will end our proof with two uses of $\forall$I.

1	(1) $\forall x \forall y(Rxy \rightarrow Ryx)$	A
2	(2) $\forall x \forall y(Rxy \leftrightarrow Sxy)$	A
	(n-2) $Sab \rightarrow Sba$	$\rightarrow$I
	(n-1) $\forall y(Say \rightarrow Sya)$	$\forall$I
	(n) $\forall x \forall y(Sxy \rightarrow Syx)$	$\forall$I

Step 2. Since our goal is now a conditional I will assume its antecedent and try to prove its consequent. After assuming Sab it is obvious that the letters to plug into line 2 are 'a' and 'b' for x and y. Then by SL I can get Rab.

1	(1) $\forall x \forall y(Rxy \rightarrow Ryx)$	A
2	(2) $\forall x \forall y(Rxy \leftrightarrow Sxy)$	A
3	(3) Sab	A
2	(4) $\forall y(Ray \leftrightarrow Say)$	2 $\forall$E
2	(5) $Rab \leftrightarrow Sab$	4 $\forall$E
2	(6) $Sab \rightarrow Rab$	5 $\leftrightarrow$E
2,3	(7) Rab	3,6 $\rightarrow$E
	(n-2) $Sab \rightarrow Sba$	$\rightarrow$I
	(n-1) $\forall y(Say \rightarrow Sya)$	$\forall$I
	(n) $\forall x \forall y(Sxy \rightarrow Syx)$	$\forall$I

Step 3. Now that we have Rab it is obvious that we plug in 'a' and 'b' to line 1. This will lead to Rba. Now the key is to realize

1	(1) $\forall x \forall y(Rxy \rightarrow Ryx)$	A
2	(2) $\forall x \forall y(Rxy \leftrightarrow Sxy)$	A
3	(3) Sab	A

that we can go back to line 2 and use this premise again. This time we have Rba and we want to get another 'S' claim. So this time I will plug in 'b' for x and 'a' for y. If I do that, it is easy to see how to get Sba and thus finish the problem.

2	(4) ∀y(Ray ↔ Say)	2 ∀E
2	(5) Rab ↔ Sab	4 ∀E
2	(6) Sab → Rab	5 ↔E
2,3	(7) Rab	3,6→E
1	(8) Rab → Rba	1∀Ex2
1,2,3	(9) Rba	7,8→E
2	(10) Rba ↔ Sba	2∀Ex2
1,2,3	(11) Sba	9,10↔P
1,2	(12) Sab → Sba	11→I(3)
1,2	(13) ∀y(Say→Sya)	12 ∀I
1,2	(14) ∀x∀y(Sxy → Syx)	13 ∀I

On Tuesday, Nov 14th I mentioned that since existential sentences are really just giant disjunctions and the order of disjuncts clearly doesn't matter, ∃x(Px v Qx) is equivalent to ∃xPx v ∃xQx. One direction you have to prove on your homework, the other direction is a bit trickier. I went through it, but we were rushed for time at the end. So here it is in full:

∃xPx v ∃xQx ├ ∃x(Px v Qx)

Step 1: It is important to note that the first premise has main connective 'v' so you can't simply plug in a letter for ∃E. In order to use line 1, you have to use vE. Since the goal is an existential, it is no help to work backwards. Since it isn't clear what to do, I will assume the opposite of the goal in order to use RAA.

1	(1) ∃xPx v ∃xQx	A
2	(2) ~∃x(Px v Qx)	A [for RAA]
	CONTRADICTION	
	(n) ∃x(Px v Qx)	RAA

Step 2: In order to use line 1 I have to use vE so I need to get the negation of one of . disjuncts. Then I can get the other side by vE and contradict that side as well. In order to get ~∃xPx I will use RAA. Since I can't use line 2 any other way, I will use it as part of my contradiction. So I will aim for its opposite.

1	(1) ∃xPx v ∃xQx	A
2	(2) ~∃x(Px v Qx)	A [for RAA]
3	(3) ∃xPx	A [for RAA]
	∃x(Px v Qx)	New Goal
	~∃xPx	RAA
	CONTRADICTION	
	(n) ∃x(Px v Qx)	RAA

Step 3: The part of the proof I need to do is basically going from line 3 to line

1	(1) ∃xPx v ∃xQx	A
2	(2) ~∃x(Px v Qx)	A [for RAA]

NEW GOAL. In fact, I have done this problem in the supplement for hwm 8. Since line 3 is an ∃x statement, I plug in a new name ‘a’ and then use it to get my new goal and then repeat the goal by ∃E. Now this contradicts line 2 as required, so I can do my RAA.

3	(3) ∃xPx	A [for RAA]
4	(4) Pa	A [for ∃E]
4	(5) Pa v Qa	4 vI
4	(6) ∃x(Px v Qx)	5 ∃I
3	(7) ∃x(Px v Qx)	3,6 ∃E(4)
2	(8) ~∃xPx	2,7RAA(3)
1,2	(9) ∃xQx	1,8 vE
	CONTRADICTION	
	(n) ∃x(Px v Qx)	RAA

Step 4: Now that I have contradicted one side of line 1, I just have to show that the other side also leads to a contradiction. And it does, for the same reasons. Since both sides lead to a contradiction, I can do my final RAA to get my goal.

1	(1) ∃xPx v ∃xQx	A
2	(2) ~∃x(Px v Qx)	A [for RAA]
3	(3) ∃xPx	A [for RAA]
4	(4) Pa	A [for ∃E]]
4	(5) Pa v Qa	4 vI
4	(6) ∃x(Px v Qx)	5 ∃I
3	(7) ∃x(Px v Qx)	3,6 ∃E(4)
2	(8) ~∃xPx	2,7RAA(3)
1,2	(9) ∃xQx	1,8 vE
10	(10) Qa	A [for ∃E]
10	(11) Pa v Qa	10 vI
10	(12) ∃x(Px v Qx)	11 ∃I
1,2	(13) ∃x(Px v Qx)	9,12 ∃E(10)
1	(14) ∃x(Px v Qx)	2,13 RAA(2)